Appendix C: Magnetic Coordinate Definitions and Nomenclature

This appendix presents definitions of various magnetic coordinates. The intent is not only to collect the definitions in one place, but also to note some of the problems involved with using them and to attempt to standardize some of the nomenclature. Of particular importance are the ambiguities introduced by parameters such as McIlwain's L_m which require the use of a magnetic moment fixed in time.

C.1 Physical "Constants"

Table 64 lists several fundamental parameters which are either constant or are often treated as constants.

Table 64. List of fundamental physical parameters.

Parameter, value, definition	Comments
$M_E \approx 8\times10^{22}$ A-m^2	Earth magnetic dipole moment. M_E can be computed for a given epoch from the IGRF coefficients using $$M_E = \frac{4\pi}{\mu_0} B_S R_E^3$$ where $$B_S = \sqrt{(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2}$$
$k_0 = \frac{\mu_0 M_E}{4\pi} = B_S R_E^3 \approx 0.31$ G-R_E^3	Earth magnetic dipole parameter. Sometimes referred to as the dipole moment.
$\mu_0 = 4\pi\times10^{-7}$ Wb/A-m	Permeability of free space
$R_E \approx 6371.2$ km	Earth mean radius. NOTE: this is the value used for the IGRF normalizations, and does not correspond to any officially recognized radius.

C.2 Mapping Transformations and Definitions

C.2.1 Magnetic field intensity *B*

One of the simplest coordinates is the magnetic field intensity B. Strictly speaking, to be used as a coordinate, this should be B_m, the intensity at the particle's mirror point. Thus, for directional data the magnetic field intensity should be mapped to the mirror point using the relation

$$\frac{\sin^2 \alpha_1}{B_1} = \frac{\sin^2 \alpha_2}{B_2} \tag{C1}$$

or

$$B_m = \frac{B_{local}}{\sin^2 \alpha} \tag{C2}$$

For omnidirectional data, or if the pitch angle is not given, we assume locally-mirroring particles, and B is simply B calculated for the ephemeris point.

C.2.2 Normalized magnetic field intensity *B/B₀*

B/B_0 is simply the magnetic field intensity above normalized by the magnetic intensity at the equator. However, there is ambiguity about the definition of B_0. Several versions of B_0 are encountered:

(1) The minimum value of B encountered on a magnetic field line. Generally this is obtained while one is integrating along the field line to calculate the second adiabatic invariant.

(2) An approximate (or average) value defined by

$$B_0 = \frac{k_0}{L^3} \tag{C3}$$

where k_0 is the Earth's magnetic dipole parameter (not the magnetic moment).

(3) An approximate (or average) value where k_0 is fixed at $0.311653\ \text{G-R}_\text{E}^3$. This is the definition used with the AE8/AP8 particle maps, and the one used by McIlwain in defining his L_m parameter.

Because of this ambiguity, it is recommended that the term B_0 be reserved for the McIlwain definition (case (3) above), *i.e.*:

$$B_0 = \frac{0.311653\ \text{G-R}_\text{E}^3}{L_m^3} \tag{C4}$$

When mapping pitch angle data to the equator or calculating the equatorial pitch angle of a particle, definition (1) above should be used. In this case, we recommend using the term $B_{\min}$ or B_{eq} to distinguish from case (3). Note, however, that there are a few cases (e.g., Shabansky orbits) where a field line may have multiple minima in B.

C.2.3 Equatorial pitch angle α_0

The equatorial pitch angle can easily be computed by

$$\sin^2 \alpha_0 = \frac{B_{\min}}{B_{local}} \sin^2 \alpha \tag{C5}$$

For locally mirroring particles (or for omnidirectional fluxes)

$$\alpha_0 = \sin^{-1} \sqrt{\frac{B_{\min}}{B_{local}}}; \quad \frac{B_{local}}{B_{\min}} = \frac{1}{\sin^2 \alpha_0} \tag{C6}$$

Again, in this case the true $B_{\min}$ should be used instead of B_0, if possible.

Another expression [*Schulz and Lanzerotti*, 1974] is:

$$\left[\frac{y}{Y(y)}\right]^2 = \frac{k_0}{R_E K^2 L^*} = \frac{\Phi}{2\pi K^2} \tag{C7}$$

where

$$y = \sin(\alpha_0^*)$$

Two approximations are available for $Y(y)$:

$$Y(y) = 2T_0(1-y) + (T_0 - T_1)(y \ln y + 2y - 2y^{1/2})$$

where

$$\begin{aligned} T_0 &= 1 + \frac{1}{2\sqrt{3}} \ln(2+\sqrt{3}) \\ &\approx 1.38017 \\ T_1 &= \frac{\pi}{6}\sqrt{2} \\ &\approx 0.74048 \end{aligned} \tag{C7a}$$

or

$$Y(y) = 2.760346 + 2.357194 y - 5.117540 y^{3/4} \tag{C7b}$$

C.2.4 Magnetic latitude λ_m

The magnetic latitude is usually calculated from the geodetic latitude, longitude, and altitude using a tilted dipole representation of the Earth's magnetic field (a magnetic longitude can also be calculated). It is not usually specified whether the magnetic latitude refers to a centered or offset dipole (which does not really make a large difference). Some formulations (*e.g.*, Pfitzer's INVARM routine) use a fixed dipole tilt angle which is strictly valid only for one particular epoch. The ONERA-DESP subroutine get_terms calculates the dipole moment at epoch (variable b0), the direction cosines of the dipole tilt (variables alpha and beta), and the eccentric dipole offset (x0, y0, z0).

Another version of the magnetic latitude, designated λ_g, is available:

$$\frac{B}{B_{\min}} = \frac{\sqrt{4-3\cos^2\lambda_g}}{\cos^6\lambda_g} \tag{C8}$$

or

$$B = \frac{M_E}{L^3}\frac{\sqrt{4-3\cos^2\lambda_g}}{\cos^6\lambda_g} \tag{C9}$$

This can be considered to be a generalized magnetic latitude which can be used with non-dipolar fields. Note that these relations define λ_g implicitly and require an inverse procedure to obtain λ_g as a function of B or B/B_0. *Roberts* [1964], as described by *Cabrera and Lemaire* [2007], provides a simple procedure for performing this calculation.

C.2.5 Invariant latitude Λ

The invariant latitude is used mainly for low altitude phenomena such as the aurora. Its definition is

$$\cos^2\Lambda = \frac{1}{L} \tag{C10}$$

C.2.6 First adiabatic invariant (magnetic moment) μ

The first adiabatic invariant captures the particle's gyration around the field line and is given by

$$\mu = \frac{p_\perp^2}{2mB} = \frac{p^2\sin^2\alpha_{spc}}{2mB_{spc}} \tag{C11}$$

where p is the particle momentum ($p_\perp$ is the momentum perpendicular to the magnetic field).

C.2.7 Second adiabatic invariant

The second adiabatic invariant captures the bounce motion of a particle along a magnetic field line. It is calculated by integrating between particle mirror points A and A', and hence is a function of the particle pitch angle. The "true" second invariant J is given by

$$J \equiv 2\int_{A}^{A'} mv_{\parallel} ds \tag{C12}$$

More widely used is the parameter I:

$$I \equiv \frac{J}{2mv} \equiv \int_{A}^{A'} \sqrt{1-\frac{B(s)}{B_m}}\, ds \tag{C13}$$

Recently the Kaufmann K parameter has become more popular:

$$\begin{aligned} K &\equiv \frac{J}{\sqrt{8m_0\mu}} = I\sqrt{B_m} \\ &= \int_{A}^{A'} \left[B_m - B(s)\right]^{1/2} ds \end{aligned} \tag{C14}$$

Again, all of these quantities require integration along a field line through some model magnetic field.

C.2.8 Third adiabatic invariant Φ

The third or flux invariant is equal to the flux of **B** enclosed by the surface J=constant and captures the drift motion of a trapped particle drifting around the Earth. It requires integrating around an entire drift shell:

$$\Phi = \oint \mathbf{dl}\cdot\mathbf{A} = \oiint_{S} \mathbf{da}\cdot\mathbf{B} \tag{C15}$$

Φ has units of G-R_E^2 or Webers (Wb).

C.2.9 Roederer's L*

In order to arrive at a parameter which would have more physical significance than Φ in labeling a particle drift shell, Roederer developed L^*, defined as

$$L^* = -\frac{2\pi k_0}{\Phi R_E} = -\frac{\mu_0 M_E}{2\Phi R_E} \tag{C16}$$

where M_E is the Earth magnetic dipole moment, μ_0 is the permeability of free space, and k_0 is the magnetic dipole parameter:

$$k_0 = \frac{\mu_0 M_E}{4\pi} \qquad \text{(C17)}$$

Note that for L^* to be an adiabatic invariant, k_0 or M_E must be assumed to be constant, *i.e.*, not functions of time.

C.2.10 McIlwain's L_m

McIlwain's L_m has become one of the most used and misunderstood parameters in magnetospheric physics. Strictly speaking, L_m labels a magnetic field line and is approximately the radial distance at which that field line intersects the magnetic equator. L_m *approximately* labels a drift shell, although this approximation gets worse as radial distance increases (Roederer's L^* *does* label a drift shell). L_m is defined as a function of B and the second invariant I:

$$L_m = f(B, I)$$

and is generally approximated using a function given by McIlwain or another developed by Hilton. Both of these functions use a constant based on the Earth dipole moment. Hilton recommended using the dipole moment for the epoch for which the calculations are being made, but this practice leads to a loss of adiabatic invariance. L_m should always be calculated with k_0=0.311653 G-R_E^3, the value originally used by McIlwain. Note that this value for k_0 corresponds to an epoch in the mid 1950s.

C.2.11 Invariant radius R_{inv} and invariant altitude h_{inv}

The invariant radius is defined by

$$R_{inv} = L_m \cos^2 \lambda_g \qquad \text{(C18)}$$

R_{inv} is approximately the radial distance of a point in space with L_m and λ_g. Polar plots of particle flux in R_{inv}-λ_g space are often used to provide a physical picture of the spatial distribution of the flux.

Closely related to R_{inv} is the invariant altitude h_{inv} discussed by *Cabrera and Lemaire* [2007]:

$$h_{inv} = (R_{inv} - 1)R_E \qquad \text{(C19)}$$

Both R_{inv} and h_{inv} are ideal for mapping fluxes at low altitudes where the flux is controlled more by neutral atmospheric density (and hence altitude) than by the magnetic field.

C.3 Nomenclature

Table 65 provides a proposed nomenclature which should be followed in order to avoid confusion and ambiguity.

Table 65. List of quantities.

B	Magnetic field intensity
B_0	Approximate minimum magnetic field intensity along a field line (C4)
B_{min}	Minimum magnetic field intensity along a field line
h_{inv}	Invariant Altitude (C19)
I	Alternate form of second adiabatic invariant (C13)
J	Omnidirectional flux
J	Second adiabatic invariant (C12)
j	Directional flux
j_{90}, j_{perp}	Directional flux perpendicular to magnetic field
K	Kaufmann K parameter, related to second adiabatic invariant (C14)
k_0	Earth magnetic dipole parameter (C17)
L^*	Roederer L parameter (C16)
L_m	McIlwain L parameter
M_E	Earth magnetic dipole moment
R_E	Earth mean radius
R_{inv}	Invariant Radius (C18)
Φ	Third adiabatic invariant (C13)
Λ	Invariant latitude (C10)
α	Particle pitch angle
α_0	Equatorial pitch angle
μ	First adiabatic invariant (particle magnetic moment) (C11)
λ_g	Generalized magnetic latitude (C8)
λ_m	Dipole magnetic latitude