Appendix B: Converting from \((\Phi, K)\) to \((L, B)\) Coordinates and Vice-versa

B.1 Introduction

Because the \(K-\Phi\) mapping has not been widely used in the past, it is very useful to be able to transform to and from \(K-\Phi\) and the more traditional \(B-L\) system. Here \(B-L\) is used generically to refer to different transforms of \(B\) (e.g., \(B, B/B_0\), etc.) and both \(L_m\) and \(L^*\).

Unfortunately, to date there is no simple transformation between \(K-h_{\text{min}}\) and \(B-L\).

This appendix discusses the conversion between \((\Phi, K)\) and \((L, B)\) coordinates. The following definitions are noted:

- \(B\) refers to \(B_m\), the magnetic field at the mirror point.
- \(B_{eq}\) is the magnetic field at the magnetic equator, also referred to as \(B_{min}\), the minimum magnetic field on the field line.
- \(B_0\) is the approximate value of \(B_{eq}\) based on the McIlwain definition – see equation (B5).
- The term \((L, B)\) means a generic coordinate system which refers specifically to the McIlwain \(L_m\). Here, \(B\) can refer to \(B_m, B_m/B_{eq},\) or \(B_m/B_0\).

We generally know either \((\Phi, K)\) or \((L, B)\). We want to be able to convert from one to the other, and we especially need to know the equatorial pitch angle \(\alpha_{eq}\):

\[
\alpha_{eq} = \sin^{-1} \sqrt{\frac{B_{min}}{B_m}}
\]

or

\[
y = \sin(\alpha_{eq}) = \sqrt{\frac{B_{min}}{B_m}}
\]

(B1)

B.2 Method 1 (Hilton)

In order to convert from \((\Phi, K)\) to \((L, B)\) coordinates or from \((L, B)\) to \((\Phi, K)\), we first note some basic definitions.

Roederer’s \(L^*\):

\[
L^* = \frac{2\pi k_0}{\Phi} = \frac{\mu_0 M_E}{2\Phi}
\]

(B2)
where $M_E$ is the Earth magnetic dipole moment, $\mu_0$ is the permeability of free space, and $k_0$ is the magnetic dipole parameter:

$$k_0 = \frac{\mu_0 M_E}{4\pi} \quad (B3)$$

Kaufmann’s $K$ [Kaufmann, 1965]:

$$K \equiv \frac{J}{\sqrt{8m_\mu \mu}} = J\sqrt{B_m} \quad (B4)$$

$$= \int_A^B [B_m - B(s)]^{1/2} \, ds$$

We also have to make a couple of approximations:

$$B_{eq} \approx B_0 = \frac{k_0}{L_m^3}; \quad k_0 \equiv 0.311653 \ \text{G-R}^3_E \quad (B5)$$

$$L_m \approx L^* \quad (B6)$$

Finally, we use the approximation of Hilton [1971] to obtain $L_m$ as a function of $B$ and $I$:

$$X = I^3 B_m / k_0 \quad (B7)$$

$$v = X^{1/3} \quad (B8)$$

$$F = \frac{L_m B_m}{k_0} = a_1 v^3 + a_2 v^2 + a_3 v + a_4 \quad (B9)$$

With these relations in hand, the procedure to go from $(\Phi, K)$ to $(L_m, B/B_0)$ is as follows:

1) Compute $L^* \approx L_m$ using (B2).

2) Knowing $L_m$ and $K$, compute $B_m$ using an iterative procedure:
   a) Assume a value of $B_m$.
   b) From $B_m$ and $K$, compute $I$ from (B4).
   c) From $I$ and $B_m$ use Hilton’s function to obtain $L_m$.
   d) Iterate on $B_m$ until $L_m$ converges on the input value.

3) Compute $B_0$ from (B5). We then know $B_m/B_0$.

To go from $(L_m, B/B_0)$ to $(\Phi, K)$ we reverse the process:
1) Knowing \( L_m \) compute \( B_0 \) from (B5), then compute \( B \).

2) Knowing \( L_m \) and \( B \), invert Hilton’s function to obtain \( I \). (NOTE: since Hilton’s function is a polynomial in \( v \), it’s relatively easy to find the roots.)

3) Knowing \( I \) and \( B \), compute \( K \).

4) Again assuming \( L^* \approx L_m \), compute \( \Phi \) using (B1).

### B.3 Method 2 (Schulz and Lanzerotti)

For a dipole field, we can get the following from Schulz and Lanzerotti [1974]:

\[
K = \frac{Y(y)}{y} \sqrt{\frac{\Phi}{2\pi}}
\]  

(B10)

where \( Y(y) \) is a function for which several approximations are available. Generally, this doesn’t do us much good by itself because we generally know \( \Phi \) and \( K \) and want to find \( y \). Paul O’Brien has provided a way to invert (B10) to get \( y \) as a function of \( \Phi \) and \( K \).

### B.4 Summary

If we know \( L_m, B_m, B_{eq}, \Phi, \) and \( K \), we’re home free: we can derive virtually any other needed property of the particles. Otherwise we need to make some assumptions or approximations.

- Method 1 relies on the approximations \( B_{eq} \approx B_0 \) and \( L_m \approx L^* \). Some quick calculations indicate that the former is good to within about 5% out to \( B_m / B_{eq} \approx 2 ; B_0 \) is generally smaller than \( B_{eq} \). The latter seems to be good to within about 5%.

- Method 2 relies on the dipole approximation

### B.5 Implementation

A package of MATLAB routines has been written to implement these procedures.

- `function Lm=hiltonBI(B, I, k0)` implements Hilton’s function to find \( L_m \) given \( B \) and \( I \) – equations (6) – (8).
- `function [I B] = findIB(K, L, k0)` implements the procedure to go from \( (\Phi, K) \) to \( (L_m, B/B_0) \).
- `function I=hiltonBL(B, Lm, k0)` implements the inverse of Hilton’s function to find \( I \) given \( L_m \) and \( B \). All that remains is to compute \( K \) from \( I \) and \( B \) and \( \Phi \) from \( L^* \approx L_m \).
- `function y = K_to_y(K, Phi)` implements Paul’s procedure to find the inverse of (9).